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ON PRIME-INDEPENDENT MULTIPLICATIVE FUNCTIONS

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1. Introduction

An arithmetical function $f : \mathbb{N} \rightarrow \mathbb{C}$ with $f(\mathbb{N}) \neq \{0\}$ is called a *multiplicative* function if $f(mn) = f(m)f(n)$ holds for any $m, n \in \mathbb{N}$ with $(m, n) = 1$. Obviously, if f is multiplicative, then $f(1) = 1$ and the values of $f(n) (n \geq 2)$ depend on $f(p^\alpha) (p \in \mathbb{P}, \alpha \in \mathbb{N})$. We say a multiplicative function $f(n)$ is *prime-independent multiplicative* function if for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$, the value of $f(p^\alpha)$ does not depend on p but only on α .

We can define the prime-independent multiplicative function in another way. Suppose $g : \mathbb{N} \rightarrow \mathbb{C}$ is any map such that $g(\mathbb{N}) \neq \{0\}$. Define

$$(1.1) \quad f(n) = \begin{cases} 1, & \text{if } n = 1, \\ \prod_{p^\alpha \parallel n} g(\alpha), & \text{if } n > 1. \end{cases}$$

Then $f(n)$ is a prime-independent multiplicative function and we say it is generated from g . Throughout this paper, we use this definition.

There are many well-known prime-independent multiplicative functions.

Example 1.1. Let $a(n)$ denote the number of non-isomorphic abelian groups with n elements. It is well-known that $a(n)$ is multiplicative and $a(p^\alpha) = P(\alpha)$ for any $p \in \mathbb{P}, \alpha \in \mathbb{N}$, where $P(\alpha)$ is the number of unrestricted partitions of α . Thus $a(n)$ is prime-independent multiplicative.

Example 1.2. The Dirichlet divisor function $d(n)$ is prime-independent multiplicative since $d(p^\alpha) = \alpha + 1$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$.

Example 1.3. Suppose $n > 1$ is an integer and write $n = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$. We say an integer u is an exponential divisor of n if

$$u = p_1^{\beta_1} \cdots p_s^{\beta_s} | n \Rightarrow \beta_j | \alpha_j (j = 1, \dots, s).$$

Let $d^{(e)}(n)$ denote the number of exponential divisors of n for $n > 1$ and $d^{(e)}(1) = 1$. Then $d^{(e)}(n)$ is prime-independent multiplicative since $d^{(e)}(p^\alpha) = d(\alpha)$ for any $p \in \mathbb{P}$ and $\alpha \in \mathbb{N}$.

The aim of this paper is to study the local density property of **integer-valued** prime-independent multiplicative functions.

Definition. If $l \geq 1$ is a fixed integer and $\{a_n\}$ is any subset of \mathbb{N} . We say $\{a_n\}$ possesses the local density d_l if the limit

$$(1.2) \quad d_l = \lim_{x \rightarrow \infty} x^{-1} \sum_{n \leq x, a_n = l} 1$$

We suppose now that $g : \mathbb{N} \rightarrow \mathbb{N}$ is any map such that $g(\mathbb{N}) \neq \{1\}$ and f is generated from g . For any fixed $l \geq 1$, define

$$F_{l,f} := \{n \in \mathbb{N} : f(n) = l\}, F_{l,f}(x) = \#\{n \in \mathbb{N} : f(n) = l, n \leq x\}.$$

When $f(n) = a(n)$, The asymptotic behaviour of $F_{l,a}(x)$ was studied by Kendall and Rankin[10], Ivić[6], Krätzel[12,13], Krätzel and Wolke[14].

The asymptotic behaviour of $F_{l,f}(x)$ for general prime-independent multiplicative functions was studied by Ivić[7]. Define

$$(1.3) \quad F_{l,f}(x) = d_{l,f}x + R(x),$$

where $d_{l,f} \geq 0$ is a constant depending only on g and l .

When $g(1) = 1$, Ivić proved that uniformly in l

$$(1.4) \quad R(x) \ll x^{1/3} \log^2 x, \text{ if } g(2) = 1,$$

$$(1.5) \quad R(x) \ll x^{1/2} e^{-A\gamma(x)}$$

if there is a prime p such that $p|g(2)$ but $p \nmid l$, where $A > 0$ is a positive constant and

$$\gamma(x) := (\log x)^{3/5} (\log \log x)^{-1/5},$$

and

$$(1.6) \quad R(x) \ll x^{1/2} (\log \log x)^{c-1} \log^{-1} x$$

where $c = \min\{B \geq 1 : p|g(2), p^B \parallel l\}$.

When $g(1) > 1$, Ivić proved that if $l = p^c l'$, $p \nmid l'$, where p is a prime divisor of $g(1)$, then

$$(1.7) \quad F_{l,f}(x) \ll x (\log \log x)^{c-1} \log^{-1} x.$$

In this paper we shall further improve Ivić's results .

2. The case $g(1) = 1$

2.1. On $F_{l,f}(x)$.

In this section we consider the case $g(1) = 1$. First introduce some definitions connected with g . Since $g(\mathbb{N}) \neq \{1\}$, there exists an integer $k \geq 2$ such that $g(1) = \dots = g(k-1) = 1$, but $g(k) > 1$. We define r_0 to be the smallest $j > k$ with $g(j) = 1$ if $1 \in \{g(n) : n > k\}$ otherwise we define $r_0 = \infty$.

Let $\mathcal{Q}_k(x)$ denote the number of k -free numbers not exceeding x . If the Riemann Hypothesis (RH) is true, then for some constant $0 < \theta_k < 1/k$ the asymptotic formula

$$(2.1) \quad \mathcal{Q}_k(x) = \frac{x}{\zeta(k)} + O(x^{\theta_k})$$

holds. For example, we can take

$$\theta_2 = 17/54 + \varepsilon, \theta_k = 7/(8k+6) + \varepsilon (3 \leq k \leq 5), \theta_6 = 67/514 + \varepsilon, \text{ etc.}$$

See Jia[9] , Graham and Pintz[2].

Remark. We always suppose that $1/\theta_k \notin \mathbb{N}$.

Now we state the results of $F_{l,f}(x)$. For $l = 1$, we have

Theorem 2.1. The asymptotic formula

$$(2.2) \quad F_{1,f}(x) = d_{1,f}x + O(x^{1/k}e^{-A\gamma(x)})$$

holds.

If RH is true , then

$$(2.3) \quad F_{1,f}(x) = d_{1,f}x + d_{2,f}x^{1/r_0} + O(x^{1/(r_0+1-r_0\theta_k)})$$

for $r_0 < 1/\theta_k$ and

$$(2.4) \quad F_{1,f}(x) = d_{1,f}x + O(x^{\theta_k}),$$

for $r_0 > 1/\theta_k$.

Remark. If $r_0 = \infty$, then $F_{1,f}$ is the set of all k -free numbers.

Now we suppose $l \geq 2$. Let r denote the smallest j such that $g(j) > 1$ and $g(j)|l$. Suppose $l = g^c(r)l'$, $c > 0$, $g(r) \nmid l'$. Obviously , $r \geq k$.

Theorem 2.2. If $r = k$, then for any fixed integer $N \geq 1$ we have

$$(2.5) \quad F_{l,f}(x) = d_{l,f}x + x^{1/k} \sum_{j=1}^N Q_j(\log \log x) \log^{-j-1} x \\ + O\left(x^{1/k}(\log x)^{-N-2}(\log \log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in t of degree not exceeding $c-1$.

If $r > k$, then

$$(2.6) \quad F_{l,f}(x) = d_{l,f}x + O(x^{1/k}e^{-A\gamma(x)}).$$

If RH is true , then (2.6) can be further improved.

Theorem 2.3. Suppose RH is true and $k < r < r_0$.

If $r < 1/\theta_k$, then for any fixed integer $N \geq 1$, we have

$$(2.7) \quad F_{l,f}(x) = d_{l,f}x + x^{1/r} \sum_{j=1}^N Q_j(\log \log x) \log^{-j} x \\ + O\left(x^{1/r}(\log x)^{-N-1}(\log \log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in t of degree not exceeding $c-1$.

If $r > 1/\theta_k$, then

$$(2.8) \quad F_{l,f}(x) = d_{l,f}x + O(x^{\theta_k}).$$

Theorem 2.4. Suppose RH is true and $r > r_0$.

If $r_0 < 1/\theta_k$, then

$$(2.9) \quad F_{l,f}(x) = d_{l,f}x + d_{l,f}^*x^{1/r_0} + O(x^{1/(r_0+1-r_0\theta_k)}).$$

If $r_0 > 1/\theta_k$, then

$$(2.10) \quad F_{l,f}(x) = d_{l,f}x + O(x^{\theta_k}).$$

Taking $f(n) = d^{(e)}(n)$, we get the following Corollary 2.1.

Corollary 2.1.

If $l = 2^{cl'}$, $c > 0$, $2 \nmid l'$, then for any fixed integer $N \geq 1$ we have

$$(2.11) \quad F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + x^{1/2} \sum_{j=1}^N Q_j(\log \log x) \log^{-j-1} x \\ + O\left(x^{1/2}(\log x)^{-N-2}(\log \log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in t of degree not exceeding $c-1$.

If $2 \nmid l$, we have

$$(2.12) \quad F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + O(x^{1/2}e^{-A\gamma(x)}).$$

If $2 \nmid l$ and RH is true, then we have

$$(2.13) \quad F_{l,d^{(e)}}(x) = d_{l,d^{(e)}}x + O(x^{\theta_2}).$$

Corollary 2.2. Suppose $g(1) = g(3) = 1$, $g(2) > 1$.

(1) If $g^c(2) \parallel l$ for some $c \geq 1$, then for any fixed integer $N \geq 1$ we have

$$(2.14) \quad F_{l,f}(x) = d_{l,f}x + x^{1/2} \sum_{j=1}^N Q_j(\log \log x) \log^{-j-1} x \\ + O\left(x^{1/2}(\log x)^{-N-2}(\log \log x)^{c-1}\right),$$

where $Q_j(t)$ is a polynomial in t of degree not exceeding $c-1$.

(2) If $g(2) \nmid l$, then

$$(2.15) \quad F_{l,f}(x) = d_{l,f}x + O(x^{1/2}e^{-A\gamma(x)}).$$

If $g(2) \nmid l$ and RH is true, then

$$(2.16) \quad F_{l,f}(x) = d_{l,f}x + d_{l,f}^*x^{1/3} + O(x^{18/55+\varepsilon}).$$

2.2. On $F_{l,f}(x+y) - F_{l,f}(x)$.

It is also interesting to study the above problem in the short interval $(x, x+y]$ with $y = o(x)$. In the case of $a(n)$, it was first proved by Ivić[8] that

$$(2.17) \quad F_{l,a}(x+y) - F_{l,a}(x) = d_{l,a}y + o(y)$$

holds for $y \geq x^{581/1744} \log x$ uniformly for $l \geq 1$. Krätzel[11] proved that (2.17) is true for $y \geq x^{11/42+\varepsilon}$ and even for $y \geq x^{2/9+\varepsilon}$ if $l \equiv \pm 1 \pmod{6}$. Li[15] proved that (2.17) is true for $y \geq x^{1/5+\varepsilon}$ uniformly for $l \geq 1$.

In the general case, we have the following Theorem.

Theorem 2.5. The asymptotic formula

$$(2.18) \quad F_{l,f}(x+y) - F_{l,f}(x) = d_{l,f}y + o(y)$$

holds for $y \geq x^{1/(2k+1)+\varepsilon}$.

3. Preliminary definitions for the case $g(1) > 1$

In this section, we shall make some further definitions for the use of the case $g(1) > 1$. Suppose \mathcal{T} is a subset of \mathbb{N} such that

$$k_0 = \min_{n \in \mathcal{T}} n \geq 2, \quad \mathcal{T} \neq k_0\mathbb{N}.$$

It is easy to prove the following Lemma 3.1.

Lemma 3.1 There exist integers $k_0 = a_0 < a_1 < \cdots < a_t < b$, $d \geq 1$ such that $\{a_0, a_1, \dots, a_t\} \subset \mathcal{T}$ and for $|u| < 1$ we have

$$(3.1) \quad 1 + \sum_{n \in \mathcal{T}} u^n = \frac{(1 - u^b)^d}{\prod_{j=0}^t (1 - u^{a_j})} \times (1 + O(|u|^{b+1})).$$

Remark. d is always 1 or 2.

Definition 3.1. Define

$$S(\mathcal{T}) := \{a_0, a_1, \dots, a_t\}, \quad k(\mathcal{T}) := b, \quad E(\mathcal{T}) := d.$$

Now we define the *primitive generating* subset of \mathcal{T} . We suppose $\mathcal{T} \setminus k_0\mathbb{N} \neq \emptyset$. Write $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$, where

$$\mathcal{T}_0 = \{n \in \mathcal{T} : k_0 | n\}, \quad \mathcal{T}_1 = \{n \in \mathcal{T} : k_0 \nmid n\}.$$

Then $\mathcal{T}_1 \neq \emptyset$.

A subset \mathcal{T}_P of \mathcal{T} is said the *primitive generating* subset if it satisfies the following conditions:

(1). Every element of \mathcal{T}_1 can be written as a linear combination of elements of \mathcal{T}_P with non-negative integral coefficients;

(2). For any $\mathcal{A} \subset \mathcal{T}_P$, $\mathcal{A} \neq \mathcal{T}_P$, there exists an $n \in \mathcal{T}_1$ such that n can't be written as a linear combination of elements of \mathcal{A} with non-negative integral coefficients.

If $k_0 | \mathcal{T}$, then define $\mathcal{T}_P = \{k_0\}$.

We also need estimates of multi-dimensional divisor functions. Suppose $S = \{u_1, u_2, \dots, u_t\}$ is a finite subset of \mathbb{N} with $u_1 < u_2 < \cdots < u_t$. The multi-dimensional divisor function $d(n; S)$ is defined by

$$d(n; S) = d(u_1, u_2, \dots, u_t; n) := \sum_{n = n_1^{u_1} \cdots n_t^{u_t}} 1.$$

Write

$$D(x; S) = \sum_{n \leq x} d(n; S) = \sum_{j \in S} c_j x^{1/j} + \Delta(x; S).$$

Now we define $\delta(S)$ to be a real number such that $0 \leq \delta(S) < 1/u_1$ and the estimate

$$\Delta(x; S) \ll x^{\delta(S)}$$

holds, and define $\delta^*(S)$ to be a real number such that $0 \leq \delta^*(S) < 1/u_1$ and the estimate

$$\sum_{x < n \leq x+y} d(n; S) = C_{u_1} y x^{1/u_1-1} (1 + o(1)) + O(x^{\delta^*(S)})$$

holds. Obviously, $\delta^*(S) \leq \delta(S)$.

It is important to get good values for $\delta(S)$, which can be done by the theory of exponential sums and the theory of the Riemann Zeta-function. The value of $\delta^*(S)$ is connected with many short interval problems. For example, the value of $\delta^*({1, k})$ ($k \geq 2$) is connected with the distribution of k -free numbers in short intervals, the value of $\delta^*({2, 3})$ is connected with the distribution of square-full numbers in short intervals, the value of $\delta^*({3, 4, 5})$ is connected with the distribution of cube-full numbers in short intervals, etc.

For more details about multi-dimensional divisor problems, see Krätzel[11].

4. The case $g(1) > 1$

4.1. On $F_{l,f}(x)$.

Now we consider the case $g(1) > 1$. Let k_0 denote the smallest element in the set $\mathcal{G}_0 = \{n \in \mathbb{N} : g(n) = 1\}$ if it is not empty; otherwise define $k_0 = \infty$. Let r denote the smallest j such that $g(j) > 1$ and $g(j) | l$. Suppose $l = g^c(r)l'$, $c > 0$, $g(r) \nmid l'$.

Theorem 4.1. If $r < k_0$, then for any $N \geq 1$ we have

$$(4.1) \quad F_{l,f}(x) = x^{1/r} \sum_{j=1}^N Q_j(\log \log x) \log^{-j} x + O\left(x^{1/r} (\log x)^{-N-1} (\log \log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in u of degree not exceeding $c-1$.

Now suppose $l = 1$ or $l \geq 2$ with $r > k_0$. We have the following Theorem 4.2.

Theorem 4.2.

(I) Suppose $l = 1$ or $l \geq 2$ with $r > k(\mathcal{G}_0)$.

If $\delta(S(\mathcal{G}_0))k(\mathcal{G}_0) > 1$, then we have

$$(4.2) \quad F_{l,f}(x) = \sum_{j \in S(\mathcal{G}_0)} c_{l,j} x^{1/j} + O(x^{\delta(S(\mathcal{G}_0))}).$$

If $\delta(S(\mathcal{G}_0))k(\mathcal{G}_0) < 1$, then we have

$$(4.3) \quad F_{l,f}(x) = \sum_{j \in S(\mathcal{G}_0)} c_{l,j} x^{1/j} + O(x^{1/k(\mathcal{G}_0)} e^{-A\gamma(x)}).$$

(II) Suppose $k_0 < r \leq k(\mathcal{G}_0)$ and let S_1 denote the set of elements in $S(\mathcal{G}_0)$ less

If $\delta(S_1)r > 1$, then

$$(4.4) \quad F_{l,f}(x) = \sum_{j \in S_1} c_{l,j} x^{1/j} + O(x^{\delta(S_1)}).$$

If $\delta(S_1)r < 1$, then for any fixed integer $N \geq 1$ we have

$$(4.5) \quad F_{l,f}(x) = \sum_{j \in S_1} c_{l,j} x^{1/j} + x^{1/r} \sum_{j=1}^N Q_j(\log \log x) \log^{-j-s_0} x \\ + O\left(x^{1/r} (\log x)^{-N-1-s_0} (\log \log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in u of degree not exceeding $c-1$, and

$$s_0 = \begin{cases} 0, & \text{if } r < k(\mathcal{G}_0), \\ 1, & \text{if } r = k(\mathcal{G}_0), E(\mathcal{G}_0) = 1, \\ 2, & \text{if } r = k(\mathcal{G}_0), E(\mathcal{G}_0) = 2. \end{cases}$$

For the function $f(n) = d(n)$, we have the following Corollary 4.1.

Corollary 4.1. Suppose $l \geq 2$. Let p denote the smallest prime divisor of l and write $l = p^c l'$, $c > 0$, $p \nmid l'$. Then for any fixed integer $N \geq 1$, we have

$$(4.6) \quad F_{l,d}(x) = x^{\frac{1}{p-1}} \sum_{j=1}^N Q_j(\log \log x) \log^{-j} x + O\left(x^{\frac{1}{p-1}} (\log x)^{-N-1} (\log \log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in u of degree not exceeding $c-1$.

Corollary 4.2. Suppose $r_0 \geq 3$ is a fixed integer such that

$$\{2, 3, \dots, r_0\} \subset \mathcal{G}_0, g(1) > 1, g(r_0 + 1) > 1.$$

If $r = 1$, then for any fixed integer $N \geq 1$, we have

$$(4.7) \quad F_{l,f}(x) = x \sum_{j=1}^N Q_j(\log \log x) \log^{-j} x + O\left(x (\log x)^{-N-1} (\log \log x)^{c-1}\right),$$

where $Q_j(u)$ is a polynomial in u of degree not exceeding $c-1$.

If $r = r_0 + 1$ ($r_0 = 3, 4, 5$), then for any fixed integer $N \geq 1$, we have

$$(4.8) \quad F_{l,f}(x) = c_{l,f}^{(2)} x^{1/2} + c_{l,f}^{(3)} x^{1/3} \\ + x^{\frac{1}{r_0+1}} \sum_{j=1}^N Q_{j,l,f}(\log \log x) \log^{-j-t_0} x + O\left(x^{\frac{1}{r_0+1}} (\log x)^{-N-1-t_0} (\log \log x)^{c-1}\right),$$

where $Q_{j,l,f}(u)$ is a polynomial in u of degree not exceeding $c-1$, and

$$t_0 = \begin{cases} 1, & \text{if } r_0 = 3, 4, \\ 2, & \text{if } r_0 = 5. \end{cases}$$

If $r > r_0 + 1$ ($r_0 = 3, 4, 5$) or $r \geq r_0 + 1$ ($r_0 \geq 6$), then

$$(4.9) \quad F_{l,f}(x) = c_{l,f}^{(2)} x^{1/2} + c_{l,f}^{(3)} x^{1/3} + O(x^{1/t_1} e^{-A\gamma(x)}),$$

where

$$t_1 = \begin{cases} 4, & \text{if } r_0 = 3, \\ 5, & \text{if } r_0 = 4, \\ 6, & \text{if } r_0 \geq 5. \end{cases}$$

4.2. $F_{l,f}(x+y) - F_{l,f}(x)$.

Now we study the short interval results. Note that if $F_{l,f} \neq \emptyset$, then l is factorizable on $g(\mathbb{N} \setminus \mathcal{G}_0)$. If l has a factorization $l = g^{c_1}(r_1) \cdots g^{c_e}(r_e)$, then let $\mathcal{G} = \{r_1, \dots, r_e\}$. And we define

$$\mathcal{G}^* = \{\mathcal{G} | l \text{ has a factorization on } g(\mathcal{G})\}.$$

Let $\mathcal{G}_1, \dots, \mathcal{G}_h$ denote all elements of \mathcal{G}^* and define

$$\mathcal{T}_j = \mathcal{G}_0 \cup \mathcal{G}_j, (j = 1, \dots, h).$$

If $k_0\mathbb{N}$ is not a subset of \mathcal{G}_0 , then we define $k_1 \geq 2$ to be an integer such that

$$\{k_0, 2k_0, \dots, (k_1 - 1)k_0\} \subset \mathcal{G}_0, k_1 k_0 \notin \mathcal{G}_0;$$

If $k_0\mathbb{N} \subset \mathcal{G}_0$, then define $k_1 = \infty$.

For $l = 1$, we have the following Theorem 4.3.

Theorem 4.3. Suppose $k_0 < \infty$. If

$$y \geq \begin{cases} x^{\frac{k_0-1}{k_0} + \frac{1}{(2k_1+1)k_0} + \varepsilon}, & \text{if } k_0 | \mathcal{G}_0, \\ x^{\frac{k_0-1}{k_0} + \max\left(\frac{1}{(2k_1+1)k_0}, \delta^*(\mathcal{G}_{0P})\right) + \varepsilon}, & \text{if } k_0 \nmid \mathcal{G}_0, \end{cases}$$

then

$$(4.10) \quad F_{1,f}(x+y) - F_{1,f}(x) = c_0 y x^{1/k_0-1} (1 + o(1)),$$

where $c_0 > 0$ is some positive constant.

Now suppose $l \geq 2$ and we have the following Theorem 4.4.

Theorem 4.4.

If $r = 1$, then for $y \geq x^{7/12+\varepsilon}$ we have

$$(4.11) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y (\log \log x)^{c-1} \log^{-1} x (1 + o(1)),$$

where c_0 is a constant.

If $1 < r < k_0$, then for

$$y \geq x^{\frac{r-1}{r} + \max\left(\frac{7}{12r}, \delta^*(\mathcal{T}_{1P}), \dots, \delta^*(\mathcal{T}_{hP})\right) + \varepsilon}$$

we have

$$(4.12) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{\frac{1}{r}-1} (\log \log x)^{c-1} \log^{-1} x (1 + o(1)).$$

If $r > k_0$, then for

$$y \geq x^{\frac{k_0-1}{k_0} + \max\left(\frac{1}{(2k_1+1)k_0}, \delta^*(\mathcal{T}_{1P}), \dots, \delta^*(\mathcal{T}_{hP})\right) + \varepsilon}$$

we have

$$(4.13) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_{k_0,f} y x^{1/k_0-1} (1 + o(1)).$$

Taking $f(n) = d(n)$, we get

Corollary 4.3.

(1) Suppose $2|l$, then the asymptotic formula

$$(4.14) \quad F_{l,d}(x+y) - F_{l,d}(x) = c_0 y (\log \log x)^{c-1} \log^{-1} x (1 + o(1))$$

holds for $y \geq x^{7/12+\varepsilon}$.

(2) Suppose $l = p_1^{c_1} \cdots p_e^{c_e}$ with $2 < p_1 < \cdots < p_e$. Let $r = p_1 - 1$, then the asymptotic formula

$$(4.15) \quad F_{l,d}(x+y) - F_{l,d}(x) = c_0 y x^{\frac{1}{r}-1} (\log \log x)^{c-1} \log^{-1} x (1 + o(1))$$

holds for $y \geq x^{\frac{r-1}{r} + \max(\frac{7}{12r}, \delta(l)) + \varepsilon}$, where

$$\delta(l) = \max_{\substack{l=n_1 \cdots n_d \\ d \geq 2, n_j > 1}} \delta^*(\{n_1 - 1, \dots, n_d - 1\}_P).$$

Corollary 4.4. Suppose $r_0 \geq 3$ is a fixed integer and $\{2, 3, \dots, r_0\} \subset \mathcal{G}_0$.

If $r = 1$, then for $y \geq x^{7/12+\varepsilon}$ the asymptotic formula

$$(4.16) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y (\log \log x)^{c-1} \log^{-1} x (1 + o(1))$$

holds.

If $r \neq 1$, then

$$(4.17) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{-1/2} (1 + o(1))$$

holds for $y \geq x^{5/8+\varepsilon}$.

Corollary 4.5. Suppose $r_0 \geq 4$ is a fixed integer such that

$$\{3, 4, \dots, r_0\} \subset \mathcal{G}_0, g(1) > 1, g(2) > 1, g(r_0 + 1) > 1.$$

If $r = 1, 2$, then for $y \geq x^{\frac{r-1}{r} + \frac{7}{12r} + \varepsilon}$, we have

$$(4.18) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{\frac{1}{r}-1} (\log \log x)^{c-1} \log^{-1} x (1 + o(1)).$$

Suppose $r_0 = 4$. If $l = 1$ or $r > 4$, then for $y \geq x^{2/3+1/11+\varepsilon}$, we have

$$(4.19) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{-2/3} (1 + o(1)).$$

Suppose $r_0 \geq 5$. If $l = 1$ or $r > r_0$, then for $y \geq x^{2/3+19/159+\varepsilon}$, we have

$$(4.20) \quad F_{l,f}(x+y) - F_{l,f}(x) = c_0 y x^{-2/3} (1 + o(1)).$$

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